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The inverse Penrose transform on Riemannian twistor spaces

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Abstract

With respect to the Dirac operator and the conformally invariant Laplacian, an explicit description of the inverse Penrose transform on Riemannian twistor spaces is given. A Dolbeault representative of cohomology on the twistor space is constructed from a solution of the field equation on the base manifold.

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0. Introduction

The Penrose transform is a method to give solutions of the Dirac equation and the conformally invariant Laplacian. It is done by relating solutions of the field equations to cohomology with values in a certain holomorphic line bundle over the twistor space of the manifold.

The Penrose transform on four-dimensional half-conformally flat manifolds was studied by Hitchin [H]. Murray [M] generalized it to higher-dimensional conformally flat manifolds.

The correspondence between cohomology and the space of solutions of the field equation was proved to be one-to-one. But the sufficiency part of the proof, that is, to construct a cohomology class from a solution of the field equation, was proved indirectly in both papers. In the four-dimensional case, an explicit formula for the inverse Penrose transform was given by Woodhouse [W].

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In this paper, we shall give an explicit description of the inverse Penrose transform for Riemannian manifolds by constructing a Dolbeault representative of the corresponding cohomology (Definition 4.4 in the four-dimensional case and Definition 2.3 in the higher-dimensional cases). Definition 4.4 is equivalent to the formula given by Woodhouse.

Let M be a $2n$-dimensional spin manifold and let V be a Hermitian vector bundle on M with a connection. We assume the conditions on the metric of M and the curvature of V which enable us to perform the Penrose transform. Then the twistor space $Z^{\pm}(M)$ of M is a complex manifold, and the hyperplane bundle H and the pull back bundle $p^{-1}V$ on $Z^{\pm}(M)$ are holomorphic bundles. Let $\Delta^{\pm}(M)$ be the spin bundle on M. Then the twisted differential form on $Z^{\pm}(M)$ which represents the cohomology class corresponding to $\phi \in \Gamma(M, S^m\Delta^{\pm}(M) \otimes V)$ is written as

$$\mathcal{Q}_m(\phi) = (n+m-2)!F^{(n+m-2)}(D)j(\phi),$$

where D is a differential operator and $F^{(n+m-2)}$ is the $(n+m-2)$th derivative of the power series $F(x) = \sum_{k=0}^{\infty} x^k/(k!)^2$. The lifting $j(\phi)$ is written as

$$j : \Gamma(M, S^m\Delta^{\pm}(M) \otimes V) \to \Gamma\,(Z^{\pm}(M), \Lambda_V^{0,(1/2)n(n-1)}Z^{\pm}(M) \otimes H^{-2n+2-m} \otimes p^{-1}V),$$

where $\Lambda_V^{0,(1/2)n(n-1)}Z^{\pm}(M)$ is a line subbundle of $\Lambda^{0,(1/2)n(n-1)}Z^{\pm}(M)$. The definitions of D and $\Lambda_V^{0,(1/2)n(n-1)}Z^{\pm}(M)$ in the four-dimensional case are slightly different from those in the higher-dimensional cases.

In the four dimensional case, $\Lambda_V^{0,1}Z^{\pm}(M)$ is the space of vertical forms with respect to the Levi-Civita connection on M, which is defined globally. The operator D is also defined as a global operator.

In the higher-dimensional cases, the assumption of the metric of M means that there are local conformally flat coordinates. The line subbundle $\Lambda_V^{0,(1/2)n(n-1)}Z^{\pm}(M)$ is the space of vertical forms corresponding to the trivialization with respect to these coordinates. The operator D is also defined with respect to them. Although j and D are defined to be local operators, it is shown that the constructed inverse Penrose transform is defined independently of the particular conformally flat coordinates which are chosen.

In both cases, it is shown that D^{n+1} vanishes. Hence the construction is in fact a finite sum.

In the proof of the vanishing of $\bar{\partial}\mathcal{Q}_m(\phi)$ and the independence of the construction of $\mathcal{Q}_m$ with respect to the coordinates chosen in the higher-dimensional cases, the defining equations of $Z^{\pm}$ as a subvariety of $\mathbf{P}(\Delta^{\pm})$ play an important role (see Lemma 2.6). They were given in [11] in the course of my definition of Riemannian twistor spaces. Although they are trivial in four- and six- dimensional cases, they are still useful by our extension of notation of multi-indices.

In Section 1, we review the theory of the Penrose transform on Riemannian twistor spaces. In Sections 2 and 3, we give the inverse Penrose transform on $2n$-dimensional manifolds with $n \geq 3$. The local construction of the inverse Penrose transform is given in Section 2, and the independence of the construction with respect to the coordinates which are chosen is proved in Section 3. In Section 4, we deal with the four-dimensional case, in which a non-conformally flat manifold may admit the integrable twistor space.

1. The Penrose transform

Let us review the Penrose transform for the Dirac operator and the conformally invariant Laplacian on even-dimensional spin manifolds (see [H]: the four-dimensional case, and [M]: the higher-dimensional cases).

Let M be a $2n$-dimensional spin manifold, and let V be a Hermitian vector bundle on M with a connection. Let $\Delta^+(M)$ (resp. $\Delta^-(M)$) be the positive (resp. negative) spin bundle. The field equations that we consider are the Dirac operator

$$d_m : \Gamma(M, S^m\Delta^{\pm}(M)\otimes V) \to \Gamma(M, \Delta^{\mp}(M)\otimes S^{m-1}\Delta^{\pm}(M)\otimes V)$$

for $m > 0$ and the conformally invariant Laplacian

$$\begin{aligned} d_0 : \Gamma(V) &\to \Gamma(V) \\ \phi &\mapsto \nabla^*\nabla(\phi) + \frac{n-1}{2(2n-1)} r\phi, \end{aligned}$$

where r is the scalar curvature of M. The differential operator d_m is conformally invariant with conformal weight $n - 1 + \frac{1}{2}m$.

Assume $n \geq 2$. Let Z^+ be the parameter space of complex structures of a $2n$-dimensional real vector space compatible with a certain metric and a certain orientation. If we consider the opposite orientation, we have a similar manifold Z^-. Let $\mathrm{SO}(M)$ be the oriented orthonormal frame bundle of M. Riemannian twistor spaces are defined as

$$Z^{\pm}(M) = \mathrm{SO}(M) \times_{\mathrm{SO}(2n)} Z^{\pm},$$

which have natural almost complex structures. We assume that M has an integrable twistor space. Let p be the projection $p : Z^{\pm}(M) \to M$. Assume that the pull back of the curvature form of V by p is a $\mathrm{End}(p^{-1}V)$-valued $(1, 1)$-form. Then the pull back bundle $p^{-1}V$ can be naturally considered to be a holomorphic vector bundle.

There is a holomorphic line bundle H on $Z^{\pm}(M)$, which is called the hyperplane bundle. Let us define the Penrose transform:

$$\mathcal{P}_m : H^{(1/2)n(n-1)}(Z^{\pm}(M), H^{-2n+2-m}\otimes p^{-1}V) \to \Gamma(M, S^m\Delta^{\pm}(M)\otimes V).$$

By restricting cohomology classes to each fiber, we have a map

$$\begin{aligned} &H^{(1/2)n(n-1)}(Z^{\pm}(M), H^{-2n+2-m}\otimes p^{-1}V) \\ &\to \bigcup_{x\in M} H^{(1/2)n(n-1)}(Z^{\pm}_x, H^{-2n+2-m})\otimes V_x. \end{aligned}$$

This induces $\mathcal{P}_m$, since $H^{(1/2)n(n-1)}(Z^{\pm}_x, H^{-2n+2-m})$ is equivalent to $S^m\Delta^{\pm}_x$ as a representation space of $\mathrm{SPIN}(T_xM)$ by the theorem of Bott–Borel–Weil–Kostant.

Theorem 1.1 ([H, Theorem 3.1; M, Theorem 32]). *The map $\mathcal{P}_m$ induces an isomorphism onto the space of the solutions of $d_m\phi = 0$.*

2. The local construction of the inverse Penrose transform

In this section, we deal with the local construction of the inverse Penrose transform when the base manifold is $2n$-dimensional with $n \geq 3$. We assume that M is an open subset of $\mathbb{R}^{2n}$ with the standard metric and the vector bundle V is trivial. Hence we can safely omit V by considering it as a trivial line bundle.

By Dolbeault's theorem, we have a representation of the cohomology group:

$$H^{(1/2)n(n-1)}(Z^{\pm}(M), H^{-2n+2-m}) = \frac{\ker \overline{\partial}|_{\Gamma(\Lambda^{0,(1/2)n(n-1)}Z^{\pm}(M)\otimes H^{-2n+2-m})}}{\overline{\partial}\Gamma(\Lambda^{0,(1/2)n(n-1)-1}Z^{\pm}(M) \otimes H^{-2n+2-m})}.$$

We construct a $\overline{\partial}$-closed form in $\Gamma(Z^{\pm}(M), \Lambda^{0,(1/2)n(n-1)}Z^{\pm}(M) \otimes H^{-2n+2-m})$ from a solution of $d_m\phi = 0$.

We begin by defining a differential operator D acting on

$$\Gamma(Z^{\pm}(M), \Lambda^{*}Z^{\pm}(M) \otimes H^{-2n+2-m}).$$

The action of SO($2n$) on $Z^{\pm}$ induces a linear map

$$\mathcal{F} : \mathfrak{so}(2n) \to \Gamma(Z^{\pm}, \Theta),$$

where Θ is the holomorphic tangent bundle on $Z^{\pm}$, which is considered to be the $(1, 0)$-part of the complexified tangent bundle $TZ^{\pm} \otimes \mathbb{C}$. With respect to the Lie algebra structures, we have

$$\mathcal{F}([a, b]) = -[\mathcal{F}(a), \mathcal{F}(b)]. \tag{2.1}$$

Let

$$e_a = \partial/\partial x_a, \quad e^a = \mathrm{d}x_a, \quad a = 1, \ldots, 2n$$

be the standard frames of TM and $T^{*}M$, respectively. Let $(E^a_b)_{1\leq a,b\leq 2n}$ be the frame of End(TM) defined as $E^a_b e_c = \delta^a_c e_b$ and put $F_{ab} = E^a_b - E^b_a$. By considering it as an element of $\mathfrak{so}(2n)$, we define a vector field $\mathcal{F}_{ab}$ on $M \times Z^{\pm} = Z^{\pm}(M)$ to be

$$\mathcal{F}_{ab} = \mathcal{F}(F_{ab}).$$

Then we define a first-order differential operator D acting on $\Gamma(Z^{\pm}(M), \Lambda^{*}Z^{\pm}(M) \otimes H^{-2n+2-m})$ as

$$D = -L_{e_a} i(\overline{\mathcal{F}_{ab}})e^b,$$

where L is the Lie derivative and i is the interior multiplication. The action of the horizontal form e^b is the exterior multiplication.

Lemma 2.1.

(1) *The differential operator D is invariant under the conformal automorphism of $\mathbb{R}^{2n}$ if it is considered as an operator on $Z^{\pm}(\mathbb{R}^{2n})$.*

(2) *Let l and l' be non-negative integers. Then we have*

$$D\Gamma(Z^{\pm}(M), \Lambda^{l,l'}Z^{\pm}(M)\otimes H^{-2n+2-m}) \subset \Gamma(Z^{\pm}(M), \Lambda^{l,l'}Z^{\pm}(M)\otimes H^{-2n+2-m})$$

(3) *We have $D^{n+1}=0$.*

Proof. We have (1) immediately by the definition. Since e^b and $L_{e_a}i(\overline{\mathcal{F}_{ab}})$ are commutative and the space of horizontal (0, 1)-forms are n-dimensional, (3) follows by (2). We will prove (2) later in this section, since it is needed a system of local coordinates of the fiber. □

Put

$$F(x)=\sum_{k=0}^{\infty}\frac{x^k}{(k!)^2}.$$

An essential property of this function is the following lemma.

Lemma 2.2. *Let l be a non-negative integer. Then we have an equation*

$$xF^{(l+2)}(x)=-(l+1)F^{(l+1)}(x)+F^{(l)}(x).$$

Remark. The function F also appears in the construction of the inverse Penrose transform of the Dolbeault complex [I2].

Let $\Lambda_V^{0,(1/2)n(n-1)}$ be the line subbundle of $\Lambda^{0,(1/2)n(n-1)}Z^{\pm}(M)$ spanned by vertical forms. If we identify H^{-1} with $\overline{H}$ by the Hermitian metric, we have

$$\Lambda^{0,(1/2)n(n-1)}Z^{\pm}(M)\otimes H^{-2n+2-m}\supset\Lambda_V^{0,(1/2)n(n-1)}\otimes H^{-2n+2-m}\simeq\overline{H}^m,$$

since $\Lambda_V^{0,(1/2)n(n-1)}\simeq\overline{H}^{-2n+2}$. By the theorem of Bott–Borel–Weil–Kostant, we have an isomorphism

$$H^0(Z^{\pm},H^m)\simeq(S^m\Delta^{\pm})^*.$$

Hence we can define a lifting

$$j:\Gamma(M,S^m\Delta^{\pm}(M))\to\Gamma(Z^{\pm}(M),\Lambda^{0,(1/2)n(n-1)}Z^{\pm}(M)\otimes H^{-2n+2-m}).$$

By using D and j, we can define the local inverse Penrose transform as follows.

Definition 2.3. We define a map

$$\begin{aligned}Q_m:\Gamma(M,S^m\Delta^{\pm}(M))&\to\Gamma(Z^{\pm}(M),\Lambda^{0,(1/2)n(n-1)}Z^{\pm}(M)\otimes H^{-2n+2-m})\\ \phi&\mapsto(n+m-2)!F^{(n+m-2)}(D)j(\phi).\end{aligned}$$

Remark. We shall prove in the following section that $\mathcal{Q}_m$ does not depend on the conformally flat coordinates of M and the trivialization of V which are chosen. Hence $\mathcal{Q}_m$ can be considered as a global conformally invariant operator.

Let us describe the Penrose transform $\mathcal{P}_m$ by using the Dolbeault representation of cohomology classes. Since the complex dimension of $Z_x^\pm$ is $(1/2)n(n-1)$, the isomorphism

$$H^{(1/2)n(n-1)}(Z_x^\pm, H^{-2n+2-m}) \simeq S^m \Delta_x^\pm$$

induces a map

$$\Gamma(Z_x^\pm, \Lambda^{0,(1/2)n(n-1)} Z_x^\pm \otimes H^{-2n+2-m}) \to S^m \Delta_x^\pm.$$

Hence we have a map

$$\tilde{\mathcal{P}}_m : \Gamma(Z^\pm(M), \Lambda^{0,(1/2)n(n-1)} Z^\pm(M) \otimes H^{-2n+2-m}) \to \Gamma(M, S^m \Delta^\pm(M)).$$

Then $\tilde{\mathcal{P}}_m$ induces the Penrose transform $\mathcal{P}_m$. Since we have $\tilde{\mathcal{P}}_m(\mathcal{Q}_m(\phi)) = \phi$ for any section $\phi \in \Gamma(M, S^m \Delta^\pm(M))$, we finish constructing the local inverse Penrose transform by the following theorem.

Theorem 2.4. *Let ϕ be a section of $S^m \Delta^\pm(M)$. Then $\mathcal{Q}_m(\phi)$ is $\bar{\partial}$-closed if $d_m \phi = 0$.*

The remainder of this section is devoted to a proof of this theorem. We prove it by computing $dF^{(n+m-2)}(D) - F^{(n+m-2)}(D)d$.

Lemma 2.5.
(1) *Put $E = dD - Dd$. Then, we have*

$$E = -L_{e_a} L_{\overline{\mathcal{F}_{ab}}} e^b.$$

(2) *Let $d_H = e^a L_{e_a}$ be the exterior differentiation to the horizontal direction, and put*

$$\Gamma = e^a \wedge e^b i(\overline{\mathcal{F}_{ab}}) \sum_c (L_{\partial/\partial x_c})^2.$$

Then, we have

$$ED - DE = -2Dd_H + \Gamma, \quad \Gamma D - D\Gamma = 0, \quad d_H D - Dd_H = 0.$$

(3) *Let $f(x)$ be a power series. Then we have*

$$df(D) - f(D)d = f'(D)E - f''(D)Dd_H + \tfrac{1}{2} f''(D)\Gamma.$$

Proof.
(1) Let Ω be the curvature form of $H^{-2n+2-m}$. Let v be a vector field on $Z^\pm(M)$. Then we have

$$[d, L_v] = -[i(v), \Omega]$$

as an operator acting on $\Gamma(Z^\pm(M), \Lambda^* Z^\pm(M) \otimes H^{-2n+2-m})$. Since $[i(e_a), \Omega] = 0$ for any integer a such that $1 \le a \le 2n$, we have the desired equation.

(2) The second equation and the third one are immediate by the definitions and the formulas

$$[L_v, i(v')] = i([v, v']), \quad [L_v, L_{v'}] = L_{[v,v']} + \Omega(v, v'),$$

where v and v' are vector fields on $Z^{\pm}(M)$. By (2.1), we compute

$$[\mathcal{F}_{ab}, \mathcal{F}_{cd}] = -\delta_{ac}\mathcal{F}_{bd} + \delta_{ad}\mathcal{F}_{bc} + \delta_{bc}\mathcal{F}_{ad} - \delta_{bd}\mathcal{F}_{ac},$$

so we have

$$[E, D] = -2Dd_H + \Gamma.$$

(3) By induction on k, we have

$$dD^k - D^k d = kD^{k-1}E - k(k-1)D^{k-1}d_H + \tfrac{1}{2}k(k-1)D^{k-2}\Gamma,$$

which completes the proof. □

By Lemmas 2.2 and 2.5(3), we have

$$dF^{(n+m-2)}(D) = F^{(n+m-2)}(D)(d - d_H) + F^{(n+m-1)}(D)(E + (n+m-1)d_H) + \tfrac{1}{2}F^{(n+m)}(D)\Gamma.$$

Since $j(\phi)$ is harmonic in the vertical direction, we have

$$F^{(n+m-2)}(D)(d - d_H)j(\phi) = 0.$$

If ϕ satisfy $d_m\phi = 0$, by Lemma 2.5(2), we have

$$\tfrac{1}{2}F^{(n+m)}(D)\Gamma j(\phi) = 0.$$

Hence we complete the proof by computing the action of E.

Let us extend notation of a multi-index of the spin module in [I1], which significantly simplifies computation as we will see below. Let $(\theta_I)_{I<(1,\dots,n)}$ be the basis of the spin module Δ defined in [I1], where $I < (1, \dots, n)$ means that I is a subsequence of the sequence $(1, \dots, n)$. We regard a multi-index I as a finite sequence of possibly duplicate elements of $\{1, \dots, 2n\}$, and for $I = (i_1, \dots, i_k)$, θ_I is defined as $\theta_I = e_{i_1} * \cdots * e_{i_k} * \theta_\emptyset$ where $*$ is Clifford multiplication. Let $(Z^I)_{I<(1,\dots,n)}$ be the dual basis of (θ_I). Then we can consider Z^I for a multi-index I in the same way. Reduction of a multi-index is performed as follows. Let I be a reduced multi-index, i.e. I is a subsequence of $(1, \dots, n)$, and let i, j be distinct integers such that $1 \le i, j \le n$. Then

$$\theta_{iiI} = -\theta_I, \qquad \theta_{ijI} = -\theta_{jiI}, \qquad \theta_{(n+i)I} = \begin{cases} \sqrt{-1}\theta_{iI} & \text{if } i \notin I, \\ -\sqrt{-1}\theta_{iI} & \text{if } i \in I. \end{cases}$$

In the dual representation, we have

$$Z^{iiI} = -Z^I, \qquad Z^{ijI} = -Z^{jiI}, \qquad Z^{(n+i)I} = \begin{cases} -\sqrt{-1}Z^{iI} & \text{if } i \notin I, \\ \sqrt{-1}Z^{iI} & \text{if } i \in I. \end{cases}$$

Then we can reduce any multi-index I to a unique reduced index $I' < (1, \dots, n)$ such that $Z^I = \pm Z^{I'}$ or $Z^I = \pm\sqrt{-1}Z^{I'}$ holds. When a multi-index I is used as a set (for example $i \in I$ and $I \cup J$), it is considered to be the set of numbers contained in the reduced form of I. Let $|I|$ be the length of the reduced form of I. Then we have

$$\Delta^+ = \langle \theta_I \mid |I| \equiv 0(2) \rangle, \qquad \Delta^- = \langle \theta_I \mid |I| \equiv 1(2) \rangle.$$

Since $Z^\pm$ is a subvariety of $\mathbf{P}(\Delta^\pm)$, Z^I can be considered as a homogeneous coordinate of $Z^\pm$. For simplicity, we regard Z^I as a zero function when $|I|$ has the parity opposite to that of projectivized spinors in which the variety lies. The defining equations of $Z^\pm$ are given in [I1]. They can be given in our notation as follows.

Lemma 2.6. *For multi-indices I and J, let $d(I, J) = (I \setminus J) \cup (J \setminus I)$. Let a and b be integers such that $1 \le a, b \le 2n$. Then we have the following relations on $Z^\pm$:*
(1) $\sum_{k \in d(I,J)} Z^{kI} Z^{kJ} = 0$.
(2) $Z^{aI} Z^{aJ} = 0$.
(3) $\sum_{k \in d(I,J)} Z^{akI} Z^{bkJ} = Z^I Z^{abJ} - Z^{abI} Z^J$.

Proof. (1) is an immediate consequence of [I1, Corollary 3.3]. Let i be an integer such that $1 \le i \le n$. Then

$$Z^{iI} Z^{iJ} + Z^{(n+i)I} Z^{(n+i)J} = \begin{cases} 2Z^{iI} Z^{iJ}, & i \in d(I, J), \\ 0, & i \notin d(I, J). \end{cases}$$

Hence (2) follows immediately by (1). We can prove (3) by simple computation using (2). □

Now we fix a multi-index I. Let $z^J = Z^J / Z^I$ for a multi-index J. We let $w_{ij} = z^{ijI}$. Then $(w_{ij})_{1 \le i < j \le n}$ are local coordinates on $U_I = \{[Z^J] \in Z^+ \cup Z^- \mid Z^I \neq 0\}$.

Lemma 2.7. *For integers i and j such that $1 \le i < j \le n$, we have*

$$\partial z^J / \partial w_{ij} = \begin{cases} z^{jiJ}, & i, j \in d(I, J), \\ 0, & \text{otherwise}. \end{cases}$$

Proof. We have the relation

$$z^J = (z^{abJ} z^{cdJ} - z^{acJ} z^{bdJ} + z^{adJ} z^{bcJ}) / z^{abcdJ} \tag{2.2}$$

by Lemma 2.6(3). Thus we can prove the lemma inductively by taking appropriate integers a, b, c, and d. □

Lemma 2.8. *The vector field $\mathcal{F}_{ab}$ is written in the local coordinates as*

$$\mathcal{F}_{ab} = \sum_{1 \le i < j \le n} \tfrac{1}{2} (z^{bjI} z^{aiI} - z^{ajI} z^{biI}) \partial / \partial w_{ij}.$$

Proof. Since the one-parameter subgroup of SPIN$(2n)$ corresponding to F_{ab} is $\cos\frac{1}{2}t + \sin\frac{1}{2}t\, e_a e_b$, its action is written as

$$Z^J(t) = \cos\tfrac{1}{2}t Z^J + \sin\tfrac{1}{2}t Z^{baJ}.$$

Thus, by using (2.2), we compute

$$\left.\frac{\mathrm{d}w_{ij}(t)}{\mathrm{d}t}\right|_{t=0} = \tfrac{1}{2}(z^{bjI}z^{aiI} - z^{biI}z^{ajI}),$$

which completes the proof. □

Lemma 2.9. *The space of horizontal* $(1,0)$*-forms on* $Z^{\pm}(M)$ *is spanned by*

$$\alpha^J = -z^{aJ}e^a, \quad J < (1,\dots,n).$$

Proof. It suffices to show the lemma when $M = \mathbb{R}^{2n}$. Its Riemannian twistor space is given in [I1, Section 8]. Let Δ' be a spin module of SPIN$(2n+2)$:

$$\Delta' = \langle \theta_J \mid J < (0,1,\dots,n)\rangle.$$

Let $Z'^{+} \subset \mathbf{P}(\Delta'^{+})$ be the parameter space of the compatible complex structure of the vector space $\mathbb{R}^{2n+2}$, which can be identified with the twistor space $Z^{+}(S^{2n})$. Since stereographic projection defines a conformal embedding $\mathbb{R}^{2n} \subset S^{2n}$, $Z^{+}(\mathbb{R}^{2n})$ is an open submanifold of $Z^{+}(S^{2n})$. Let (Z^J) be the homogeneous coordinates with respect to (θ_J). Then we have:

$$Z^{+}(\mathbb{R}^{2n}) = \{(Z^J)_{J<(0,\dots,n)} \in Z'^{+} \mid \exists J < (1,\dots,n) \text{ such that } Z^J \neq 0\}.$$

Since a translation of $\mathbb{R}^{2n}$ is a conformal transformation, it induces a holomorphic transformation of $Z^{+}(\mathbb{R}^{2n})$, which is representable by an element of SPIN$(2n+2;\mathbb{C})$. Let $x = (x_1,\dots,x_{2n})$ be an element of $\mathbb{R}^{2n}$. Then the corresponding element of SPIN$(2n+2;\mathbb{C})$ is:

$$\alpha(x) = 1 + \frac{1}{2}\sum_{a=1}^{2n} x_a e_a(\sqrt{-1}e_0 + e_{0'}),$$

where we think the standard basis of $\mathbb{R}^{2n+2}$ to be $(e_0, e_1, \dots, e_n, e_{0'}, e_{n+1}, \dots, e_{2n})$. A point on the fiber over $0 \in \mathbb{R}^{2n}$ is written as $\sum_{I \not\ni 0} Z^I\theta_I$, thus its image by the transformation $\alpha(x)$ is written as

$$\alpha(x)\sum_{J\not\ni 0} Z^J\theta_J = Z^J\theta_J + \sqrt{-1}\sum_{a=1}^{2n} x_a Z^{aJ}\theta_{0J}.$$

Hence, for a multi-index J such that $0 \notin J$, the homogeneous function

$$Z^{0J} = \sqrt{-1}\sum_{a=1}^{2n} x_a Z^{aJ}$$

is holomorphic, which completes the proof on $Z^{+}(M)$. The proof on $Z^{-}(M)$ is done in the same way. □

Remark. By the definition of the almost complex structure of twistor spaces, the lemma is true for any spin manifold M and any oriented orthonormal frame (e^a) of T^*M.

Now we can calculate the action of E. Let $I_1, \ldots, I_m$ be multi-indices having the same parity of length. Let $\theta_{I_1,\ldots,I_m} \in S^m \Delta^{\pm}$ be the symmetrization of $\theta_{I_1} \otimes \cdots \otimes \theta_{I_m}$. Then we define

$$\overline{s^{I_1,\ldots,I_m}} = j(\theta_{I_1,\ldots,I_m}).$$

We compute its action on $M \times U_I$ for a multi-index I. If $m \geq 1$, we assume $|I| \equiv |I_1|(2)$, since $U_I = \emptyset$ in other cases.

Lemma 2.10. *We have*

$$(E + (n+m-1)d_H)j(\phi) = -\frac{m}{2}\frac{\partial \phi^{abI_1,I_2,\ldots,I_m}}{\partial x_a} e^b \wedge \overline{s^{I_1,\ldots,I_m}} - \frac{2n-2+m}{2N}\frac{\partial \phi^{I_1,\ldots,I_m}}{\partial x_a}\overline{z^{aJ}}\alpha^J \wedge \overline{s^{I_1,\ldots,I_m}},$$

where $N = \sum_{J \not\ni 0} |z^J|^2$ is the Hermitian metric of H^{-1} with the standard trivialization on $M \times U_I$.

Proof. Let a and b be distinct integers such that $1 \leq a, b \leq 2n$. Then we claim

$$L_{\overline{\mathcal{F}_{ab}}}\overline{s^{I_1,\ldots,I_m}} = \frac{1}{2}\left(-\frac{2n-2+m}{N}\sum_J z^{bJ}\overline{z^{aJ}} + \sum_{i=1}^{m}\frac{\overline{z^{baI_i}}}{\overline{z^{I_i}}}\right)\overline{s^{I_1,\ldots,I_m}}. \tag{2.3}$$

Let $\overline{\rho^J}$ be the section of $\overline{H}$ corresponding to θ_J. Let $\overline{K_I}$ be the standard trivialization of $\Lambda_V^{0,(1/2)n(n-1)}$ over $M \times U_I$. Then, by the definition of j, we can write

$$\overline{s^{I_1,\ldots,I_m}} = \overline{K_I} \otimes \overline{\rho^I}^{2n-2} \otimes \overline{\rho^{I_1}} \otimes \cdots \otimes \overline{\rho^{I_m}}.$$

First, we compute

$$\begin{aligned}
L_{\overline{\mathcal{F}_{ab}}}\overline{\rho^I} &= \nabla_{\overline{\mathcal{F}_{ab}}}\overline{\rho^I} = -\frac{\overline{\mathcal{F}_{ab}}(N)}{N}\overline{\rho^I} \\
&= -\sum_{1\leq i\neq j\leq n}\frac{\overline{z^{bjI}z^{aiI}}}{2N}\left(\sum_{J \text{ s.t. } i,j\in d(I,J)} z^J\overline{z^{jiJ}}\right)\overline{\rho^I} \quad \text{[By Lemma 2.7]} \\
&= -\frac{1}{2N}\sum_J\sum_{j\in d(I,J)}\overline{z^{bjI}z^{ajJ}}z^J\overline{\rho^I} \quad \text{[By Lemma 2.6(3)]} \\
&= \frac{1}{2}\left(-\frac{1}{N}\sum_J z^{bJ}\overline{z^{aJ}} + \overline{z^{baI}}\right)\overline{\rho^I} \quad \text{[By Lemma 2.6(3)].}
\end{aligned}$$

Hence, for another multi-index I' such that $|I'| \equiv |I|(2)$, we have

$$L_{\overline{\mathcal{F}_{ab}}}\overline{\rho^{I'}} = \frac{1}{2}\left(-\frac{1}{N}\sum_J z^{bJ}\overline{z^{aJ}} + \frac{\overline{z^{baI'}}}{\overline{z^{I'}}}\right)\overline{\rho^{I'}}.$$

Second, we compute

$$L_{\overline{\mathcal{F}_{ab}}}\overline{K_I} = -(n-1)\overline{z^{baI}K_I}.$$

Therefore we have Eq. (2.3).

By Lemma 2.5(1) and (2.3), we compute

$$\begin{aligned} Ej(\phi) &= -\sum_{a\neq b}\frac{1}{2}\frac{\partial\phi^{I_1,\ldots,I_m}}{\partial x_a}e^b\left(-\frac{2n-2+m}{N}\sum_J z^{bJ}\overline{z^{aJ}} + \sum_{i=1}^{m}\frac{\overline{z^{baI_i}}}{\overline{z^{I_i}}}\right)\overline{s^{I_1,\ldots,I_m}} \\ &= -\frac{m}{2}\frac{\partial\phi^{abI_1,I_2,\ldots,I_m}}{\partial x_a}e^b\wedge\overline{s^{I_1,\ldots,I_m}} - \frac{2n-2+m}{2N}\frac{\partial\phi^{I_1,\ldots,I_m}}{\partial x_a}\overline{z^{aJ}}\alpha^J\wedge\overline{s^{I_1,\ldots,I_m}} \\ &\quad -(n-1+m)d_H j(\phi). \end{aligned}$$

Hence we complete the proof. □

Proof of Lemma 2.1(2). We have

$$D = \frac{1}{2}\sum_{1\le i<j\le n} i(\partial/\partial\overline{w_{ij}})\left(\overline{z^{aiI}\alpha^{jI}} - \overline{z^{ajI}\alpha^{iI}}\right)L_{e_a}.$$

Since $L_{e_a}(\mathrm{d}w_{ij}) = 0$, $L_{e_a}(\alpha^J) = 0$ and $L_{e_a}(\overline{\rho^J}) = 0$, we complete the proof. □

Since D and α^J are commutative, the second term of Lemma 2.10 can be neglected modulo $(1,0)$-forms. Hence, if $m = 0$, we finish the proof of the theorem. If $m > 0$, the coefficient of the first term is that of $\theta_{bI_1}\otimes\theta_{I_2,\ldots,I_m}$ in $\frac{1}{2}md_m(\phi)$. Hence, if $d_m(\phi)$ vanishes, we have

$$(E + (n+m-1)d_H)j(\phi) \equiv 0 \quad \text{modulo } (1,0)\text{-forms}$$

for any non-negative integer m. Thus we complete the proof of the theorem.

3. Well-definedness of $\mathcal{Q}_m$ as a global operator

By Theorem 2.4, we prove that Definition 2.3 gives an inverse Penrose transform locally when the base manifold is conformally flat. In this section we show that the construction is independent with respect to the conformally flat coordinates which are chosen and gives the global inverse Penrose transform.

We continue assuming that the base manifold is $2n$-dimensional with $n \ge 3$, hence the metric of the base manifold is conformally flat. We have a local inverse Penrose transform $\mathcal{Q}_m$ by Definition 2.3 on each chart which has conformally flat coordinates. By a theorem of Liouville [DFN Theorem 15.2], a coordinate transformation is a certain restriction of an orientation preserving conformal automorphism of S^{2n} fixing the point which is regarded as the center of each chart. The orientation preserving conformal automorphism group of S^{2n} is $SO_0(1, 2n+1)$. Let G be the isotropy subgroup at $0 \in \mathbb{R}^{2n} \subset S^{2n}$. Then we shall show the invariance of $\mathcal{Q}_m$ under the action of G near the origin.

Let $CE(2n)$ be the orientation preserving conformal automorphism group of $\mathbb{R}^{2n}$. Since a local conformal map can be extended uniquely to a conformal automorphism of S^{2n}, $CE(2n)$ can be considered to be a subgroup of $SO_0(1, 2n+1)$. Let $CO(2n)$ be the isotropy subgroup of $CE(2n)$ at the origin. Let τ be the conformal map defined as

$$\begin{aligned} \tau : \mathbb{R}^{2n} \setminus \{0\} &\to \mathbb{R}^{2n} \setminus \{0\} \\ x &\mapsto x/|x|^2. \end{aligned}$$

By computing the Lie algebra of G, we can show that the group G is generated by $CO(2n)$ and $\tau \circ T(x) \circ \tau$ for $x \in \mathbb{R}^{2n}$, where $T(x)$ is the translation map on $\mathbb{R}^{2n}$ by x. Since D and j are invariant under the action of $CE(2n)$, $\mathcal{Q}_m$ is invariant under the action of $CO(2n)$. It is also invariant under $T(x)$ for any $x \in \mathbb{R}^{2n}$, so we can show its invariance under $\tau \circ T(x) \circ \tau$ on $\mathbb{R}^{2n} \setminus \{0, \tau(-x)\}$ by showing its invariance under τ on $\mathbb{R}^{2n} \setminus \{0\}$. This proves the invariance of $\mathcal{Q}_m$ under $\tau \circ T(x) \circ \tau$ near the origin, since $\mathcal{Q}_m(\phi)$ is expressible as a polynomial of jets of ϕ.

Hence it suffices to show the invariance of $\mathcal{Q}_m$ under τ on $\mathbb{R}^{2n} \setminus \{0\}$. The following lemma is used to reduce the computation to a certain point on the fiber over the point $x_0 = {}^t(1, 0, \dots, 0)$ instead of computing it on the whole space $Z^{\pm}(\mathbb{R}^{2n} \setminus \{0\})$.

Lemma 3.1. *The group* $CO(2n)$ *acts transitively on* $Z^{\pm}(\mathbb{R}^{2n} \setminus \{0\})$.

Proof. Since $CO(2n)$ acts transitively on $\mathbb{R}^{2n} \setminus \{0\}$, it suffices to prove the transitivity on the fiber over the point x_0. The isotropy subgroup at x_0 contains the subgroup which is naturally identified with $SO(2n-1)$. Then it acts on $Z^{\pm}_{x_0}$, and the isotropy subgroup at $(x_0, [\theta_I])$ for an appropriate I is naturally identified with $U(n-1)$. Thus the isomorphism

$$SO(2n-1)/U(n-1) \simeq SO(2n)/U(n)$$

means that $SO(2n-1)$ acts transitively on the fiber $Z^{\pm}_{x_0}$. □

Let B be an endomorphism defined as

$$B = -[D, |x|^2],$$

where $|x|^2$ is considered to be an operator by the multiplication.

Lemma 3.2. *The operators* D *and* B *are commutative.*

Proof. We fix a multi-index I. Let z^J and w_{ij} be as in Section 2. Then, by Lemmas 2.8 and 2.6(2), we have

$$[D, B] = -\frac{1}{2}\sum_{a\neq b,d}\sum_{i\neq j}\sum_{k\neq l}\overline{z^{bjI}z^{aiI}z^{dlI}z^{akI}}\,i(\partial/\partial\overline{w_{ij}})e^b i(\partial/\partial\overline{w_{kl}})e^d$$

$$= \frac{1}{2}\sum_{b}\sum_{d}\sum_{i\neq j}\sum_{k\neq l}\overline{z^{bjI}z^{dlI}(z^{biI}z^{bkI} + z^{diI}z^{dkI})}\,i(\partial/\partial\overline{w_{ij}})e^b i(\partial/\partial\overline{w_{kl}})e^d,$$

where we put $\partial/\partial w_{ji} = -\partial/\partial w_{ij}$ for integers i and j such that $1 \le i < j \le n$. Since, for fixed integers b and d, we have

$$\sum_{i\neq j}\overline{z^{bjI}z^{biI}}\,i(\partial/\partial\overline{w_{ij}}) = 0, \qquad \sum_{k\neq l}\overline{z^{dlI}z^{dkI}}\,i(\partial/\partial\overline{w_{kl}}) = 0,$$

we complete the proof. □

We can relate $\tau^*(j(\phi))$ and $j(\tau^*\phi)$ by using B as follows.

Lemma 3.3. *We have* $\tau^*(j(\phi)) = \exp(|x|^{-2}B)j(\tau^*\phi)$.

Proof. Let κ be the orientation reversing isometry defined as

$$\kappa\left({}^{t}(x_1, x_2, \ldots, x_{2n})\right) = {}^{t}(-x_1, x_2, \ldots, x_{2n}).$$

Let G' be the transformation group generated by κ and $CO(2n)$. Then we have
(1) an element of G' preserves j and $(1/|x|^2)B$,
(2) $\tau G' = G'\tau$,
(3) G' acts on $Z^+(\mathbb{R}^{2n}\setminus\{0\}) \cup Z^-(\mathbb{R}^{2n}\setminus\{0\})$ transitively.
Hence it suffices to compute them at the point $z_0 = (x_0, [\theta_\emptyset])$. The map between the twistor spaces induced by τ is written in the homogeneous coordinates as

$$\tau : Z^+(\mathbb{R}^{2n}\setminus\{0\}) \to Z^-(\mathbb{R}^{2n}\setminus\{0\})$$
$$(x, [Z^I]) \mapsto \left(\frac{x}{|x|^2}, [x_a Z^{aI}]\right).$$

Hence $\tau(x_0, [\theta_\emptyset]) = (x_0, [\theta_1])$. We take the two systems of local coordinates

$$w_{ij} = Z^{ij}/Z^{\emptyset} \quad \text{on } U_\emptyset,\ w'_{ij} = Z^{ij1}/Z^1 \quad \text{on } U_1, \qquad 1 \le i < j \le n.$$

Then we have

$$\tau^*\overline{dw'_{ij}}\Big|_{z_0} = \begin{cases} -\overline{dw_{1j}} + e^j - \sqrt{-1}e^{n+j}, & i = 1,\\ \overline{dw_{ij}}, & i > 1.\end{cases}$$

For each j, we have

$$\left((e^j - \sqrt{-1}e^{n+j})i(\partial/\partial\overline{w_{1j}})\right)^2 = 0.$$

Thus we have

$$\prod_{j=2}^{n}\left(1 - (e^j - \sqrt{-1}e^{n+j})i(\partial/\partial\overline{w_{1j}})\right) = \exp\left(|x|^{-2}B\right)\Big|_{z_0}.$$

Hence

$\tau^* \Lambda_V^{0,(1/2)n(n-1)} Z^-(\mathbb{R}^{2n} \setminus \{0\})\Big|_{z_0}$

is spanned by

$$\exp\left(|x|^{-2}B\right)\overline{dw_{12}} \wedge \cdots \wedge \overline{dw_{n-1,n}}.$$

Since we have $\tau^*(j(\phi)) \equiv j(\tau^*\phi)$ modulo horizontal forms, we complete the proof. □

Lemma 3.4.

(1) *We have* $D\overline{s^{I_1,\dots,I_m}} = 0$.

(2) *Put* $\overline{t^{I_1,\dots,I_m}} = |x|^{2(n+m-1)} j(\tau^*\theta_{I_1,\dots,I_m})$. *Then we have* $D\overline{t^{I_1,\dots,I_m}} = 0$.

Proof. Since $L_{e_a}\overline{s^{I_1,\dots,I_m}} = 0$ for an integer a such that $1 \le a \le 2n$, we have (1).

Let J be a multi-index. By using Lemma 2.6(2), we have

$$[D, x_a\overline{z^{aJ}}] = \frac{1}{2}\sum_b \sum_{i \ne j} \overline{z^{bjI} z^{biI} z^{bJ}}\, i(\partial/\partial\overline{w_{ij}})e^b = 0.$$

The mth symmetric spin bundle $S^m\Delta^{\pm}(\mathbb{R}^{2n} \setminus \{0\})$ with conformal weight $n - 1 + \frac{1}{2}m$ is transformed by τ as follows.

$$\tau^*\theta_{I_1,\dots,I_m} = c|x|^{-2(n+m-1)} x_{a_1} \cdots x_{a_m} \theta_{a_1 I_1,\dots,a_m I_m},$$

where c is the constant number determined by spin structures of both ends of τ. Then

$$\overline{t^{I_1,\dots,I_m}} = c\left(x_{a_1}\overline{z^{a_1 I_1}}\right)\cdots\left(x_{a_m}\overline{z^{a_m I_m}}\right)\overline{s^{\emptyset,\dots,\emptyset}}.$$

Hence we complete the proof of (2). □

We have $\tau CO(2n) = CO(2n)\tau$. We have also that elements of $CO(2n)$ preserve $\mathcal{Q}_m$. Hence, by Lemma 3.1, it suffices to show that $\mathcal{Q}_m$ is invariant under τ at a certain point on the fiber over x_0. Actually, we do not need to specify a special point on the fiber, so we compare two inverse Penrose transforms on the fiber over the point x_0. For simplicity, we write $x' = (x_2, \dots, x_{2n})$. Since they are linear with respect to ϕ, it suffices to show $\tau^*(\mathcal{Q}_m(\phi)) = \mathcal{Q}_m(\tau^*\phi)$ with respect to the section

$$\phi = {x_1}^l f(x')\theta_{I_1,\dots,I_m},$$

where l is a non-negative integer and $f(x')$ is a homogeneous polynomial of degree l'.

Lemma 3.5.

(1) *Let a be an integer such that* $1 \le a \le 2n$. *Then* $[D, x_a]$ *and* D *are commutative.*

(2) *At points of the fiber over* x_0, $D^{l'} f(x')$ *is an endomorphism of the vector bundle, and the pull back by* τ *is computed as*

$$\tau^*\left(D^{l'} f(x')\Big|_{p^{-1}(x_0)}\right) = D^{l'} f(x').$$

(3) *The pull back of B by τ is computed as*

$$\tau^* B\big|_{p^{-1}(x_0)} = -B.$$

Proof. We have

$$[D, x_a] = -i(\mathcal{F}_{ab})e^b. \tag{3.1}$$

Since this is constant with respect to $x_1, \ldots, x_{2n}$, we prove the assertion (1). The Jacobian matrix of $\tau : \mathbb{R}^{2n} \setminus \{0\} \to \mathbb{R}^{2n} \setminus \{0\}$ is

$$|x|^{-2}R(x),$$

where $R(x)$ is the reflection with respect to the hyperplane with the normal vector x. Since the operator (3.1) is transformed as a one-form, we have

$$\tau^*[D, x_a]\big|_{p^{-1}(x_0)} = \begin{cases} -[D, x_1], & a = 1, \\ [D, x_a], & a > 1. \end{cases}$$

Hence we have the equation of (2). Since $B|_{p^{-1}(x_0)} = -2[D, x_1]$, we also have the equation of (3). □

If $k < l'$, then we have $D^k f(x') = 0$ on the fiber over x_0. Hence, by Lemma 3.5(2), we have

$$\tau^*\left(\mathcal{Q}_m(\phi)\right) = \frac{(n+m-2)!}{l'!} D^{l'} f(x')\tau^*\left(F^{(n+m-2+l')}(D){x_1}^l\overline{s^{I_1,\ldots,I_m}}\right).$$

On the other hand, since ${x_1}^l f(x')$ is homogeneous of degree $l + l'$, we have

$$\tau^*\phi = |x|^{-2(l+l')}{x_1}^l f(x')\tau^*\theta_{I_1,\ldots,I_m}.$$

In the same way, we have

$$\mathcal{Q}_m(\tau^*\phi) = \frac{(n+m-2)!}{l'!} D^{l'} f(x')F^{(n+m-2+l')}(D)|x|^{-2(n+m-1+l+l')}{x_1}^l\overline{t^{I_1,\ldots,I_m}}.$$

Hence we can show the invariance of $\mathcal{Q}_m$ under τ by the following lemma.

Lemma 3.6. *Let n' be a non-negative integer. At points on the fiber over x_0, we have*

$$\tau^*\left(F^{(n')}(D){x_1}^l\overline{s^{I_1,\ldots,I_m}}\right) = F^{(n')}(D)|x|^{-2(n'+1+l)}{x_1}^l\overline{t^{I_1,\ldots,I_m}}.$$

Proof. We prove this by induction on l. Let $l = 0$. By Lemmas 3.4(1) and 3.3, we have

$$n'!\,\tau^*\left(F^{(n')}(D)\overline{s^{I_1,\ldots,I_m}}\right) = \exp(B)j(\tau^*\theta_{I_1,\ldots,I_m}).$$

On the other hand, by Lemmas 3.2 and 3.4(2), we have

$$n'!\,F^{(n')}(D)|x|^{-2(n'+1)}\overline{t^{I_1,\ldots,I_m}} = \exp(B)j(\tau^*\theta_{I_1,\ldots,I_m}).$$

Hence we have the equation when $l = 0$.

Let us assume that the equation is satisfied for integers less than or equal to l. Then

$$\begin{aligned} F^{(n')}(D){x_1}^{l+1}\overline{s^{l_1,\dots,l_m}} &= \sum_{k=0}^{\infty}\frac{1}{k!(n'+k)!}D^k x_1 {x_1}^l\overline{s^{l_1,\dots,l_m}} \\ &= \sum_{k=0}^{\infty}\frac{1}{k!(n'+k)!}\left(-\frac{k}{2}BD^{k-1}+D^k\right){x_1}^l\overline{s^{l_1,\dots,l_m}} \\ &= -\tfrac{1}{2}BF^{(n'+1)}(D){x_1}^l\overline{s^{l_1,\dots,l_m}} + F^{(n')}(D){x_1}^l\overline{s^{l_1,\dots,l_m}}. \end{aligned}$$

Hence, by Lemma 3.5(3) and the hypothesis, we have

$$\begin{aligned} &\tau^*\left(F^{(n')}(D){x_1}^{l+1}\overline{s^{l_1,\dots,l_m}}\right) \\ &\quad = \tfrac{1}{2}BF^{(n'+1)}(D)|x|^{-2(n'+2+l)}{x_1}^l\overline{t^{l_1,\dots,l_m}} + F^{(n')}(D)|x|^{-2(n'+1+l)}{x_1}^l\overline{t^{l_1,\dots,l_m}}. \end{aligned}$$

Since

$$\begin{aligned} &F^{(n')}(D)(x_1-|x|^2)|x|^{-2(n'+2+l)}{x_1}^l\overline{t^{l_1,\dots,l_m}} \\ &\quad = \tfrac{1}{2}BF^{(n'+1)}(D)|x|^{-2(n'+2+l)}{x_1}^l\overline{t^{l_1,\dots,l_m}}, \end{aligned}$$

we complete the proof. □

Hence we have the equation $\mathcal{Q}_m(\tau^*\phi) = \tau^*(\mathcal{Q}_m(\phi))$ for any section ϕ. Thus we have proved the remark after Definition 2.3.

4. The inverse Penrose transform over four-manifolds

In this section, as an extension of Definition 4.4, we give the inverse Penrose transform over a four-manifold. It is an interpretation of the formula given by Woodhouse.

Let M be a four dimensional spin manifold, and let V be a Hermitian vector bundle on M with a connection. Since reversing the orientation of M exchanges $Z^+(M)$ and $Z^-(M)$, it suffices to consider the inverse Penrose transform only on $Z^+(M)$. Thus we assume that the Riemannian metric of M is anti-self-dual. Assume also that the connection of V is anti-self-dual. This means that the pull back $p^{-1}V$ can be naturally considered to be a holomorphic vector bundle with a holomorphic connection on the complex manifold $Z^+(M)$.

Let us define a differential operator D. In conformally flat case, we have local conformally flat coordinates, which significantly simplify the computation. But we do not have such coordinates on anti-self-dual manifolds. So we shall define D by using an arbitrary local orthonormal frame of TM.

Let (e_a) be an oriented local orthonormal frame of TM on an open subset U. Let (e^a) be the dual frame of T^*M. Since the twistor space is a fiber bundle over M associated to the orthonormal frame bundle, we can define horizontal tangent vectors by the Levi-Civita connection. Therefore, we consider (e_a) as a local frame of the space of horizontal vector fields. Also (e^a) is regarded as a local frame of the space of horizontal forms. Let ω be the connection form of TU with respect to the Levi-Civita connection. For each integer a such that $1 \leq a \leq 2n$, we define a differential operator acting on $\Gamma(Z^+(U), \Lambda^* Z^+(U) \otimes H^{-2-m} \otimes p^{-1}V)$ by

$$\hat{L_{e_a}} = L_{e_a} + i(e_b)\omega_a^b.$$

Then, the following lemma can be proved by simple computations.

Lemma 4.1. *Let $e'_a = e_b h_a^b$ be another local orthonormal frame. Then we have $\hat{L_{e'_a}} = \hat{L_{e_b}} h_a^b$.*

With respect to the local trivialization (e_a), we define a $(1, 0)$-vector field $\mathcal{F}_{ab}$ on $Z^+(U)$ by using the SO(4)-action on $Z^+(U)$ as in Section 2. The next lemma is an immediate consequence of the definition.

Lemma 4.2. *Let (e'_a) be as above. Let $\mathcal{F}'_{ab}$ be the vector filed corresponding to the frame (e'_a). Then we have $\mathcal{F}'_{ab} = h^{-1}{}^a_c \mathcal{F}_{cd} h_b^d$.*

Let us define an operator D acting on $\Gamma(Z^+(U), \Lambda^* Z^+(U) \otimes H^{-2-m} \otimes p^{-1}V)$ as follows

$$D = -\hat{L_{e_a}} i(\overline{\mathcal{F}_{ab}})e^b.$$

Then we have the following lemma by Lemmas 4.1 and 4.2.

Lemma 4.3. *The operator D is defined independently of the choice of a local orthonormal frame of TU. Hence it is considered to be a global operator acting on $\Gamma(Z^+(M), \Lambda^* Z^+(M) \otimes H^{-2-m} \otimes p^{-1}V)$.*

The Levi-Civita connection on the base manifold defines the decomposition of the cotangent bundle of the twistor space. Hence, as in Section 2, we define a map:

$$j : \Gamma(M, S^m \Delta^+(M) \otimes V) \to \Gamma(Z^+(M), \Lambda^{0,1} Z^+(M) \otimes H^{-2-m} \otimes p^{-1}V).$$

Since the decomposition is global in this case, the map j is also global.

Definition 4.4. A map Q_m is defined as

$$Q_m : \Gamma(M, S^m \Delta^+(M) \otimes V) \to \Gamma(Z^+(M), \Lambda^{0,1} Z^+(M) \otimes H^{-2-m} \otimes p^{-1}V)$$
$$\phi \mapsto j(\phi) + \frac{1}{m+1} Dj(\phi).$$

It can be shown by straightforward computation that this differential form is equivalent to the inverse Penrose transform given by Woodhouse [W, Section 5] when V is a trivial line bundle. Computations in [W] can be easily extended to the case of non-trivial bundle V. Hence we have the following theorem.

Theorem 4.5 ([W]). *Let ϕ be a solution of $d_m\phi = 0$, then $\bar{\partial}\mathcal{Q}_m(\phi) = 0$. Hence $\mathcal{Q}_m$ gives the inverse Penrose transform.*

Remarks.

(1) The transform $\mathcal{Q}_m$ does not depend on the Riemannian structure but the conformal structure of M. Hence the above definition is equal to Definition 4.4 in the case of a flat vector bundle over a conformally flat four-dimensional manifold.

(2) Since we have $D^2 j(\phi) = 0$, we can write

$$\mathcal{Q}_m(\phi) = m!F^{(m)}(D)j(\phi),$$

as in the higher-dimensional cases.

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